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# SLIGHTLY LARGER THAN A GRAPH $C^*$-ALGEBRA

BY

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ABSTRACT

To certain higher rank Cuntz algebras $\mathcal{O}_n$ including the classical cases we trace a certain partial isometry $U$ in its strong closure. Adjoining $U$ to $\mathcal{O}_n$ we obtain a kind of uniqueness property for this larger $C^*$-algebra. Its explanation is not entirely "Cuntz's uniqueness argumentation".

## 1. Introduction

In [B1] we introduced certain higher rank Cuntz algebras $\mathcal{O}_{\{n_1,n_2,\ldots,n_d\}}$ for pairwise relative prime integers $n_1,\ldots,n_d \geq 2$. Actually they are so-called higher rank or graph algebras [RS1, RS2, RS3, KP1, KP2, KP3], and one may also notice the tracks [P, FM] or [B2].

If the "rank" $d$ is 1 then $\mathcal{O}_n = \mathcal{O}_{\{n_1\}}$ is just the classical Cuntz algebra [C] generated by $n_1$ isometries. Making this introduction readable for a preferably large audience we shall suppose that $d = 1$. In fact the following results hold also for the higher rank Cuntz algebras with minor adaption.

So let $\mathcal{O}_n$ be the classical Cuntz algebra [C] generated by $n \geq 2$ isometries $S_0, S_1, \ldots, S_{n-1} \in B(\mathcal{H})$. Then in this paper we determine a partial isometry $U \in B(\mathcal{H})$ such that

$$(1) \qquad US_0 = S_1, US_1 = S_2, \ldots, US_{n-2} = S_{n-1} \text{ and } US_{n-1} = S_0 U.$$

Among possibly several candidates one chooses $U$ with minimal support projection and considers the $C^*$-algebra $C^*(S,U)$ generated by $\mathcal{O}_n \cup \{U\}$. Let $J$ be its

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smallest ideal such that $U+J$ is unitary in the quotient $C^*(S,U)/J$. Then the latter quotient $C^*$-algebra — denoted by $\widetilde{\mathcal{O}}_n$ — contains $\mathcal{O}_n$ canonically ("letter by letter") and is canonically unique in the sense that it does not depend on the representation $S_i \in B(\mathcal{H})$, Theorem 3.3.

A second and probably more striking uniqueness variant is this. Consider $\mathcal{O}_n$ represented on a Hilbert space $\mathcal{H}$ and let $U$ be a unitary satisfying (1). Now if $U$ is unique in the sense that it is the only operator in $B(\mathcal{H})$ with property (1), then such a situation produces canonically always the same $C^*$-algebra $\widetilde{\mathcal{O}}_n$ obtained by adjoining $U$ to $\mathcal{O}_n$, Corollary 3.4.

(It remains an open question to us whether the "uniqueness of $U$" here is really necessary.)

In contrast, these uniqueness properties cannot be explained by the anyway very general concept in [B2], say. Indeed it is straight from the equations (1) that $U^{n-1}$ has to be "zero-balanced", $\mathrm{bal}(U^{n-1}) = 0$, [B2]. But $U^{n-1}$ does not lie in a finite-dimensional $C^*$-algebra and property (B) [B2] fails.

This exotic example may allude — at least theoretically — to an extended uniqueness potential than seen up to now in the Cuntz–Krieger genre ([CK] is classical).

## 2. The partial isometry $U$

In [B1] we introduced a certain higher rank Cuntz $C^*$-algebra $\mathcal{O}_n = \mathcal{O}_{\{n_1,\dots,n_d\}}$ which is generated by certain isometries $S_{i,j} \in B(\mathcal{H})$. In the beginning of section 2 of [B1] these isometries are explained in detail and also certain notions are introduced. We shall adopt this setting and the notions completely and would like to repeat them briefly.

Let $d \geq 1$ be an integer (corresponding to the "rank" of the algebra) and $n = (n_1,\dots,n_d) \in \mathbb{N}^d$ be a vector with mutually relative prime integer entries $n_i \geq 2$. For a Hilbert space $\mathcal{H}$ we are given $d$ sets of isometries $\{S_{i,0}, S_{i,1},\dots,S_{i,n_i-1}\} \subseteq B(\mathcal{H})$ $(i = 1,\dots,d)$ which satisfy the so-called Cuntz property

$$S_{i,0}S_{i,0}^* + S_{i,1}S_{i,1}^* + \cdots + S_{i,n_i-1}S_{i,n_i-1}^* = I.$$

So the $C^*$-algebra generated by the set $\{S_{i,0}, S_{i,1},\dots,S_{i,n_i-1}\}$ is the classical Cuntz algebra $\mathcal{O}_{n_i}$ [C]. Assume that the generating isometries interact in the following way (sometimes referred as "permutation rules"):

$$S_{i,x}S_{j,y} = S_{j,X}S_{i,Y} \quad \text{whenever } x + yn_i = X + Yn_j.$$

Clearly for each tuple $(x,y) \in \{0,\dots,n_i-1\} \times \{0,\dots,n_j-1\}$ we find a unique corresponding tuple $(X,Y) \in \{0,\dots,n_j-1\} \times \{0,\dots,n_i-1\}$. One can deduce from this setting that the following "permutation rules" must also hold:

$$S_{i,x}^* S_{j,y} = S_{j,X} S_{i,Y}^* \quad \text{whenever } x + Xn_i = y + Yn_j.$$

Once again, for each given tuple $(x, y)$ one finds a unique tuple $(X, Y)$. The $C^*$-algebra generated by all isometries $\{S_{i,j} \in B(\mathcal{H}) | \ 1 \le i \le d, 0 \le j \le n_i - 1\}$ is denoted by $\mathcal{O}_n = \mathcal{O}_{\{n_1,\ldots,n_d\}}$ and does not depend on the representation on the Hilbert space; see [B1], Theorem 2.8.

Let $\mathcal{A} = \{(i,j) | \ i = 1,\ldots,d \text{ and } j = 0,\ldots,n_i - 1\}$ be our alphabet which corresponds to the given generators $S_a$, $a \in \mathcal{A}$. Then $V = \bigcup_{m\ge 0} \mathcal{A}^m$ denotes all finite words in this alphabet. But very often we will use the extraordinary subset of "ordered" words

$$W = \{((i_1,j_1),\ldots,(i_m,j_m)) \in V | \ m \ge 0 \text{ and } i_1 \le i_2 \le \cdots \le i_m\}.$$

Let $\alpha = (a_1,\ldots,a_m) \in V$. Then we write $S_\alpha = S_{a_1} S_{a_2} \cdots S_{a_m}$. Further, we associate an integer vector $|\alpha| := (N_1,\ldots,N_d) \in \mathbb{Z}^d$ to $\alpha$, where

$$N_i = \text{card}\{k \in \{1,\ldots,m\} | \ a_k = (i,j) \text{ for some } j\}.$$

For $\alpha = ((i_1,j_1),\ldots,(i_m,j_m)) \in V$ $((i_k,j_k) \in \mathcal{A})$ we introduce the integer

$$Z(\alpha) = \sum_{k=1}^{m} j_k n_{i_1} \cdots n_{i_{k-1}}.$$

This is the number which is associated to the digit representation $(j_1,\ldots,j_m)$ with respect to the basis $(n_{i_1},\ldots,n_{i_m})$, where $j_1$ is the least significant digit. Given $k \in \mathbb{Z}_+^d$ we abbreviate the product $n_1^{k_1} n_2^{k_2} \cdots n_d^{k_d}$ by $n^k$. One should then observe that for all $\alpha \in V$ we have

$$0 \le Z(\alpha) < n_{i_1} n_{i_2} \cdots n_{i_m} = n^{|\alpha|}.$$

So far the "dimension" or "rank" $d \in \mathbb{N}$ is finite. However, we shall permit also $d = \infty$ in the sequel. That means that in this case $\mathcal{O}_n$ is generated by infinitely many isometries $S_{i,j}$, $i \in \mathbb{N}$, $0 \le j < n_i$, where $n = (n_1, n_2, \ldots)$ is a sequence.

It is no problem to adopt the above notions to the infinite rank case. For example, the set of words $V$ remains the set of words, whether the alphabet is finite or infinite. But one point deserves explanation, namely we put, when $d = \infty$,

$$\mathbb{Z}_+^\infty := \{(x_1, x_2, \ldots) \in (\mathbb{Z}_+)^{\mathbb{N}} | \ x_i = 0 \text{ for all but finitely many } i \ge 1\}.$$

As also noted in [B1], an ordered word $\alpha \in W$ is uniquely determined by $|\alpha|$ and $Z(\alpha)$. More precisely, the following map $t$ is a bijection:

$$t\colon W \to \{(k,j)|\ k \in \mathbb{Z}_+^d, 0 \le j < n^k\} : t(\alpha) = (|\alpha|, Z(\alpha)).$$

(In principle, such a correlation appears in [BJ], section 2, too.)

Using this bijection and given $k \in \mathbb{Z}_+^d$ we put $\min(k) := \alpha$ for the unique $\alpha \in W$ such that $|\alpha| = k$ and $Z(\alpha) = 0$. Analogously we use $\max(k) := \alpha$ for the unique $\alpha \in W$ such that $|\alpha| = k$ and $Z(\alpha) = n^k - 1$.

Hence $S_{\min(k)}$ consists of isometries $S_{i,0}$ only, whereas $S_{\max(k)}$ consists of isometries of the form $S_{i,n_i-1}$ only.

For given $\gamma \in W$ we write $\gamma + 1 := \alpha$ where $\alpha \in W$ is uniquely determined by $|\alpha| = |\gamma|$ and $Z(\alpha) = Z(\gamma) + 1$ modulo $n^{|\alpha|}$.

Recall that the least significant digit in a word $\alpha \in W$ is the leftmost one, so, for example, $((1,3), a_2, a_3, \ldots) + 1 = ((1,4), a_2, a_3, \ldots)$. If we had here an overflow in the first digit then we would get $((1,3),(5,1),a_3,\ldots) + 1 = ((1,0),(5,2),a_3,\ldots)$.

In the sequel we shall often abbreviate $\sum_{\gamma \in W\, |\gamma|=k} X(\gamma)$ by $\sum_{|\gamma|=k} X(\gamma)$, where $k \in \mathbb{Z}_+^d$, so in such sums the index $\gamma$ is always understood to be taken from $W$.

Let $\mathcal{O}_n \subseteq B(\mathcal{H})$ be the $C^*$-algebra that is generated by the isometries $S_{i,j} \in B(\mathcal{H})$. In what follows we shall trace a partial isometry $U$ in the strong closure of $\mathcal{O}_n$ which obeys the following rules for all $1 \le i \le d$:

$$\text{(A)} \quad \begin{cases} US_{i,j} = S_{i,j+1} & \text{for } 0 \le j < n_i - 1, \\ US_{i,n_i-1} = S_{i,0}U \end{cases}$$

$$\text{(B)} \quad \begin{cases} U^*S_{i,j} = S_{i,j-1} & \text{for } 0 < j \le n_i - 1, \\ U^*S_{i,0} = S_{i,n_i-1}U^* \end{cases}$$

$$\text{(C)} \quad I = U^*U + P_{\max(\infty)},$$

$$\text{(D)} \quad I = UU^* + P_{\min(\infty)}.$$

Here $P_{\max(\infty)}$ and $P_{\min(\infty)}$ denote the limits of the nets of projections $S_{\max(N)}S^*_{\max(N)}$ and $S_{\min(N)}S^*_{\min(N)}$, respectively ($N \in \mathbb{Z}_+^d$, $N$ tends to "infinity").

In the next lemma we deduce some useful algebraic relations from properties (A)–(B). One should notice here that $Z(\alpha + 1) = 0$ simply means $\alpha = \max(k)$ for $k = |\alpha|$.

Lemma 2.1: *From* (A)–(B) *follows* ($\alpha \in W$)

$$\begin{array}{lll} US_\alpha = S_{\alpha+1}, & U^*S_{\alpha+1} = S_\alpha & \textit{if } Z(\alpha+1) \neq 0, \\ US_\alpha = S_{\alpha+1}U, & U^*S_{\alpha+1} = S_\alpha U^* & \textit{if } Z(\alpha+1) = 0, \end{array}$$

*and* $S_\alpha = U^{Z(\alpha)} S_{\min(|\alpha|)}$ *for all* $\alpha \in V$.

*Proof:* We demonstrate here the first asserted column. Indeed, by the induction hypothesis let $US_\alpha = S_{\alpha+1}$ be valid for all $\alpha \in W$ with fixed length $|\alpha| = k$ and $Z(\alpha+1) \neq 0$, and $US_\alpha = S_{\alpha+1}U$ if $Z(\alpha+1) = 0$ respectively.

Let $\beta = \alpha \circ (i,j)$ be the word $\alpha$ added by the letter $(i,j)$. One should note here that

$$Z(\beta) = Z(\alpha) + jn^{|\alpha|}.$$

Then $US_\beta = US_\alpha S_{i,j}$ is either $S_{\alpha+1}S_{i,j}$ if $Z(\alpha+1) \neq 0$ and we get $US_\beta = US_{\beta+1}$; or we get, if $Z(\alpha+1) = 0$,

$$US_\beta = S_{\alpha+1}US_{i,j} = S_{\alpha+1}S_{i,j+1}$$

by property (A) and once again, since $Z(\alpha) = n^{|\alpha|} - 1$, we have $US_\beta = US_{\beta+1}$. ∎

LEMMA 2.2: *Define* $P_\alpha = S_\alpha S_\alpha^*$ *for* $\alpha \in W$. *Particularly we have* ($k \in \mathbb{Z}_+^d$)

$$P_{\min(k)} = S_{\min(k)} S_{\min(k)}^*, \quad P_{\max(k)} = S_{\max(k)} S_{\max(k)}^*. \tag{2}$$

*Both nets* (2) *are decreasing and converge to* $P_{\min(\infty)}$ *and* $P_{\max(\infty)}$, *respectively.*

*Proof:* Notice that per definition of the $S_{i,j}$ [B1] we have $S_{i,0}S_{j,0} = S_{j,0}S_{i,0}$ and $S_{i,n_i-1}S_{j,n_j-1} = S_{j,n_j-1}S_{i,n_i-1}$ for all $1 \leq i,j \leq d$. Thus both sequences (2) are decreasing for increasing $k \in \mathbb{Z}_+^d$. ∎

LEMMA 2.3: *The properties* (A) *and* (C) *determine* $U$ *uniquely.*
*All* $U$ *satisfying* (A) *yield the same operator* $U(I - P_{\max(\infty)})$.

*Proof:* We shall use the left column of Lemma 2.1. As demonstrated above it can be proved by just using (A) and so for what follows we use (A) only.

Since $US_\alpha S_\alpha^* = S_{\alpha+1}S_\alpha^*$ by Lemma 2.1, $U$ is uniquely determined on the range projections $P_\alpha = S_\alpha S_\alpha^*$ whenever $Z(\alpha+1) \neq 0$. Recall that

$$\sum_{|\alpha|=k} P_\alpha = I \tag{3}$$

for arbitrary $k \in \mathbb{Z}_+^d$ (see [B1]). Hence the operator $U(I - P_{\max(k)})$ is unique, hence also its strong limit $U(I - P_{\max(\infty)})$. Since we assume (C) we have $UP_{\max(\infty)} = 0$ and $U$ is unique. ∎

PROPOSITION 2.4: *There exists a partial isometry $U$ in the strong closure of $\mathcal{O}_n$ satisfying* (A)–(D).

*Proof:*

STEP 1: We can already see from the previous proof how $U$ should be defined.

We define $UP_\alpha = S_{\alpha+1}^* S_\alpha$ as long as $Z(\alpha+1) \neq 0$. For a concatenated word $\alpha \circ \beta$ we have, if $|\alpha| = |\gamma|$,

$$P_{\alpha\circ\beta} \leq P_\gamma \quad \text{if } \gamma = \alpha,$$
$$P_\gamma P_{\alpha\circ\beta} = 0 \quad \text{if } \gamma \neq \alpha.$$

So we should prove whether $U$ is well defined since we have per definition two possibilities for $P_{\alpha\circ\beta} \leq P_\alpha$. But we are consistent since for a concatenated word $\alpha \circ \beta$ such that $Z(\alpha+1) \neq 0$ and $Z(\alpha \circ \beta + 1) \neq 0$ we have

$$US_{\alpha\circ\beta} = S_{\alpha\circ\beta+1} = S_{(\alpha+1)\circ\beta}, \quad \text{or}$$
$$US_{\alpha\circ\beta} = US_\alpha S_\beta = S_{\alpha+1} S_\beta = S_{(\alpha+1)\circ\beta}.$$

Hence the operator $U(I - P_{\max(k)})$ is defined for all $k \in \mathbb{Z}_+^d$, recall (3). In other words, $U$ is defined on the subspace

$$\mathcal{H}_0 = \{\xi \in \mathcal{H} |\ \exists k \in \mathbb{Z}_+^d : P_{\max(k)}\xi = 0\}$$

and we extend $U$ isometrically to $\overline{\mathcal{H}_0}$ and let $U$ be zero on the complement $\mathcal{H}_0^\perp$ to get the final $U$.

STEP 2: For all $k \in \mathbb{Z}_+^d$ we put

$$U_k := U(I - P_{\max(k)}) = \sum_{|\alpha|=k, Z(\alpha+1)\neq 0} S_{\alpha+1} S_\alpha^* \in \mathcal{O}_n$$

and it is clear from the left equivalence that $U_k$ converges strongly to $U$.

STEP 3: We shall prove property (A). First we have $US_{i,j} = S_{i,j+1}$ per definition of $U$ for a letter $(i,j)$ and as long as $j \neq n_i - 1$.

For the next computation we note the following. Let $k \in \mathbb{Z}_+^d$, $1 \leq i \leq d$ and identify the delta function for the $i$-th coordinate by $\delta_i \in \mathbb{Z}_+^d$. Then for each $\alpha \in W$ with $|\alpha| = k + \delta_i$ we find unique $0 \leq j < n_i$ and unique $\beta \in W$ with $|\beta| = k$ such that $\alpha \equiv (i,j) \circ \beta$ (see [B1] Lemma 2.1), and vice versa, uniquely determined by the formula

$$Z(\alpha) = j + Z(\beta) n_i,$$

and consequently $S_\alpha = S_{(i,j)\circ\beta} = S_{(i,j)}S_\beta$ by [B1] Lemma 2.1.

Watching this fact we can compute

$$\begin{aligned} U_{k+\delta_i} S_{i,n_i-1} &= \sum_{|\alpha|=k+\delta_i, Z(\alpha+1)\neq 0} S_{\alpha+1}S_\alpha^* S_{i,n_i-1} \\ &= \sum_{|\beta|=k, Z(\beta+1)\neq 0} S_{i,0}S_{\beta+1}S_\beta^* S_{i,n_i-1}^* S_{i,n_i-1} \\ &= S_{i,0}U_k. \end{aligned}$$

Now we let $k$ tend to infinity to obtain $US_{i,n_i-1} = S_{i,0}U$ by Step 2.

Step 4: Property (C) is valid since per definition the support projection $U^*U$ is that one which projects onto $\overline{\mathcal{H}_0}$. But $P_{\max(\infty)}$ projects onto $\mathcal{H}_0^\perp$.

Step 5: Property (B) follows from (A) and (C) by

$$U^*S_{i,j} = U^*US_{i,j-1} = (I - P_{\max(\infty)})S_{i,j-1} = S_{i,j-1}$$

if $j > 1$. But if $j = 0$ then we compute very similarly as in Step 3 ($k \in \mathbb{Z}_+^d$)

$$\begin{aligned} U_{k+\delta_i}^* S_{i,0} &= \sum_{|\alpha|=k+\delta_i, Z(\alpha+1)\neq 0} S_\alpha S_{\alpha+1}^* S_{i,0} \\ &= \sum_{|\beta|=k, Z(\beta+1)\neq 0} S_{i,n_i-1}S_\beta S_{\beta+1}^* S_{i,0}^* S_{i,0} \\ &= S_{i,n_i-1}U_k^* \end{aligned}$$

and let $k$ tend to infinity.

Step 6: Property (D) can be seen as follows. Let $P_\alpha = S_\alpha S_\alpha^*$ for $\alpha \in W$. By (C) we get $\lim_{N\in\mathbb{Z}_+^d} UP_{\max(N)} = UU^*UP_{\max(\infty)} = 0$. By Lemma 2.1 we have $UP_\alpha U^* = P_{\alpha+1}$ as long as $Z(\alpha+1) \neq 0$. So with (3)

$$\begin{aligned} UU^* &= \lim_{N\in\mathbb{Z}_+^d} \sum_{|\alpha|=N} UP_\alpha U^* = \lim_{N\in\mathbb{Z}_+^d} \sum_{|\alpha|=N, Z(\alpha+1)\neq 0} P_{\alpha+1} \\ &= \lim_{N\in\mathbb{Z}_+^d} (I - P_{\min(N)}) = I - P_{\min(\infty)}. \qquad \blacksquare \end{aligned}$$

## 3. The algebras $\widetilde{\mathcal{O}}_n$

In this section, if nothing else is mentioned, $U$ denotes the unique operator of Proposition 2.4 satisfying properties (A)–(D). In the sequel we shall denote by $C^*(S) \cong \mathcal{O}_n \subseteq B(\mathcal{H})$ the $C^*$-algebra which is generated by the isometries

$S_{i,j} \in B(\mathcal{H})$, and by $C^*(S,U)$ the $C^*$-algebra generated by the $S_{i,j} \in B(\mathcal{H})$ together with $U \in B(\mathcal{H})$.

The analogous (smaller) $*$-algebras generated by these generators are denoted by $\mathrm{Alg}^*(S)$ and $\mathrm{Alg}^*(S,U)$, respectively.

In the next lemma we shall divide out an ideal $J$ in $C^*(S,U)$ such that the resulting quotient converts the partial isometry $U$ to a unitary.

Lemma 3.1: *Let $J \subseteq C^*(S,U)$ be the closed ideal generated by $I - U^*U$ and $I - UU^*$. Then the canonical map $\pi\colon C^*(S) \to C^*(S,U)/J$ is injective.*

*Proof:*

Step 1: Assume $\pi$ were not injective. Then $J \cap C^*(S) \neq \{0\}$ is a nonzero ideal in the simple $C^*$-algebra $C^*(S)$, see [B1] Theorem 3.3, and we have $I \in J$.

Consider the algebraic ideal $J_0$ generated by $P = I - U^*U$ and $Q = I - UU^*$ in the $*$-algebra $\mathrm{Alg}^*(S,U)$. Then we have $\overline{J_0} = J$ for the norm closure in $C^*(S,U)$.

We shall show in the next "Step 2" that for all $z \in J_0$ there exists a word $\alpha \in W$ such that $zS_\alpha = 0$. Therefore, since $\|I - z\| < 1$ for some $z \in J_0$,

$$\|S_\alpha\| = \|(I - z)S_\alpha\| < 1,$$

a contradiction.

Step 2: Let $Y \in J_0$ be a word in the letters $S_{i,j}, U, P = I-U^*U, Q = I-UU^*$ and their involuted letters. So $Y$ contains at least either $P$ or $Q$. We shall find $\alpha \in W$ such that $YS_\alpha = 0$. (For arbitrary $Y \in J_0$ we achieve the same by successively cancelling each summand.)

Assume by the induction hypothesis that we have already found $\beta \in V$ such that $YS_\beta = Y_2T$ where $T$ (or sometimes $\tilde{T}$) denotes a product in the letters $S_{i,j}$ only, and $Y_2$ is a shorter word than $Y$. Let $y$ be the rightmost letter of $Y_2$, i.e., $Y_2 = \tilde{Y}y$.

If $y = S_{i,j}$ then we simply will put it to $T$ and obtain $Y_2T = \tilde{Y}\tilde{T}$. If $y = S_{i,j}^*$ then by multiplying $Y_2T$ from the right with the letter $S_{i,0}$ and "permuting" (using the Definitions of the $S_{i,j}$ [B1]) this letter to $y$ we obtain $Y_2TS_{i,0} = \tilde{Y}\tilde{T}$ or zero (using the fact that $S_{i,j}^*S_{i,k}$ is either 0 or $I$).

If $y = U$, then successively applying the rules stated in Lemma 2.1 we obtain $Y_2T = \tilde{Y}\tilde{T}$ or $\tilde{Y}\tilde{T}U$. In the latter case we multiply from the right with $S_{1,0}$ and get $\tilde{Y}\tilde{T}S_{1,1}$. Similarly we act in the case $y = U^*$.

Now if $y = P$ then we multiply from the right with $S_{1,0}$. Then we certainly have $Y_2TS_{1,0} = 0$ and at the latest we are done here. This is because we can move $P$ to the right by $PS_{i,n_i-1} = S_{i,n_i-1}P$ (use Lemma 2.1) or we have zero by $PS_{i,j} = 0$ whenever $0 \leq j < n_i - 1$. Similarly, if $y = Q$ then we multiply with $S_{1,n_1-1}$. ∎

The following proposition approximates the norm of elements in $C^*(S,U)/J$ by those of $\mathcal{O}_n$ (which has the uniqueness property).

Proposition 3.2: *If $X \in \mathrm{Alg}^*(S,U)$ then we find a net of projections $Q_N \to Q_\infty$, $Q_N \in \mathrm{Alg}^*(S)$, $N \in \mathbb{Z}_+^d$, such that $X(I - Q_N) \in \mathrm{Alg}^*(S)$ and*

$$\|X + J\|_{C^*(S,U)/J} = \|X(I - Q_\infty)\|_{C^*(S,U)} = \lim_{N \in \mathbb{Z}_+^d} \|X(I - Q_N)\|_{C^*(S)}.$$

*Neither the choice of the net $(Q_N) \subseteq \mathcal{O}_n$ nor $X(I - Q_N) \in \mathcal{O}_n$ depends on the representation on the Hilbert space $\mathcal{H}$.*

*Proof:*

Step 1: Let $X$ be a word in the letters $S_{i,j}, U$ and their involuted letters. Then by using the rules of properties (A) and (B) each occurrence of both $U$ and $U^*$ in $X$ can be moved to the right or is absorbed by a right-handed letter $S_{i,j}$. This is not so obvious for expressions $US_{i,j}^*$. But here we argue by using (B)

$$\begin{aligned} US_{i,j}^* &= (S_{i,j}U^*)^* = ((U^*)^{n_i-1-j} S_{i,n_i-1} U^*)^* \\ &= ((U^*)^{n_i-1-j} U^* S_{i,0})^* = S_{i,0}^* U^{n_i-j}. \end{aligned}$$

Similarly, $U^*$ can "skip" the involuted letter $S_{i,j}^*$.

Step 2: So it is clear that each element $X \in \mathrm{Alg}^*(S,U)$ can be written as the sum $X = \sum_{i=1}^m \lambda_i X_i U_i$ where $\lambda_i \in \mathbb{C}$, $X_i \in \mathrm{Alg}^*(S)$ and $U_i$ is a word in the two letters $\{U, U^*\}$.

Let $M \in \mathbb{N}$ be the maximal length of all words $U_i$. Let $P_\alpha := S_\alpha S_\alpha^*$ for $\alpha \in W$.

Then for $N \in \mathbb{Z}_+^d$ and $\alpha \in W$ with $|\alpha| = N$ and $M < Z(\alpha) < n^N - M$ we find a certain $\beta_i \in W$ with $|\beta_i| = N$ (see the rules of Lemma 2.1) such that

$$XP_\alpha = \sum_{i=1}^m \lambda_i X_i U_i S_\alpha S_\alpha^* = \sum_{i=1}^m \lambda_i X_i S_{\beta_i} S_\alpha^* \in C^*(S). \tag{4}$$

Step 3: Since $\lim_{N \in \mathbb{Z}_+^d} P_{\max(N)} = I - U^*U \in J$ by property (C), we get by Lemma 2.1 for all integers $k \geq 0$

$$U^{*k}(I - U^*U)U^k = \lim_{N \in \mathbb{Z}_+^d} U^{*k} P_{\max(N)} U^k = \lim_{N \in \mathbb{Z}_+^d} P_{\max(N)-k} \in J.$$

Similarly, we get $\lim_{N\in Z_+^d} P_{\min(N)+k} \in J$ by considering $I - UU^* \in J$. Let

$$Q_N := \sum_{k=-M}^{M} P_{\min(N)+k} \tag{5}$$

and $Q_\infty = \lim_{N\in\mathbb{Z}_+^d} Q_N$ strongly. (Note that $P_{\min(N)-1} = P_{\max(N)}$, $P_{\min(N)-2} = P_{\max(N)-1}$ etc., per definition, say.) Then by the above computations we have $Q_\infty \in J$.

Step 4: Abbreviate $A := C^*(S,U)$. For arbitrary $\varepsilon > 0$ we find some normalized $\xi \in \mathcal{H}$ and $N_0 \in \mathbb{Z}_+^d$ such that for all $N \geq N_0$

$$\begin{aligned}
\|X + J\|_{A/J} &\leq \|X - XQ_\infty\|_A \\
&\leq \|X(I - Q_\infty)\xi\|_{\mathcal{H}} + \varepsilon \\
&\leq \|X(I - Q_N)\xi\|_{\mathcal{H}} + \|X(Q_N - Q_\infty)\xi\|_{\mathcal{H}} + \varepsilon \\
&\leq \|X(I - Q_N)\|_A + 2\varepsilon \\
&= \|X(I - Q_N) + J\|_{A/J} + 2\varepsilon \qquad (6)\\
&\leq \|X + J\|_{A/J} + 2\varepsilon.
\end{aligned}$$

The equality (6) is due to Lemma 3.1 because by (3), (4) and (5) we have

$$X(I - Q_N) = X \sum_{|\alpha|=N, M<Z(\alpha)<n^N-M} P_\alpha \in C^*(S).$$

Considering the above proof we note that we have only used the properties (A)–(B) of $U$ to determine $Q_N$ and $X(I - Q_N)$, and the formal shape of $X(I - Q_N)$ in the letters $S_{i,j}$ only does not depend on the representation on the Hilbert space $\mathcal{H}$. (Also note [B1], the uniqueness Theorem 2.8.)

We complete the proof by varying $\varepsilon > 0$. ∎

Theorem 3.3 (Uniqueness variant 1): *Consider the higher rank Cuntz algebra $\mathcal{O}_n = \mathcal{O}_{\{n_1,n_2,\ldots,n_d\}}$ [B1] ($d \leq \infty$) generated by the isometries $S_{i,j} \in B(\mathcal{H})$. (If $d = 1$ then this is the classical Cuntz algebra [C].) Then we can find a partial isometry $U \in B(\mathcal{H})$ satisfying* (A)–(D). *We adjoin $U$ to $\mathcal{O}_n$ and obtain a $C^*$-algebra $C^*(S,U) \subseteq B(\mathcal{H})$. Next we divide out the ideal $J \subseteq C^*(S,U)$ generated by $I - U^*U$ and $I - UU^*$ to obtain the quotient $\widetilde{\mathcal{O}}_n := C^*(S,U)/J$. Then we have $\mathcal{O}_n \subseteq \widetilde{\mathcal{O}}_n$ canonically.*

*Now if we are given another representation of $\mathcal{O}_n$ generated by the isometries $\widehat{S}_{i,j} \subseteq B(\widehat{\mathcal{H}})$ and consider the procedure there, then the canonical map*

$$\sigma\colon C^*(S,U)/J \to C^*(\widehat{S},\widehat{U})/\widehat{J} : \sigma(S_{i,j} + J) = \widehat{S}_{i,j} + \widehat{J}, \sigma(U + J) = \widehat{U} + \widehat{J}$$

*is a well defined $*$-isomorphism.*

*Proof:* The theorem is a corollary of Proposition 3.2. If $X \in \mathrm{Alg}^*(S,U)$ and $\widehat{X}$ is the formally translated word, then by Proposition 3.2 we get

$$\|X + J\|_{C^*(S,U)/J} = \|X(I - Q_N)\|_{C^*(S)},$$
$$\|\widehat{X} + \widehat{J}\|_{C^*(\widehat{S},\widehat{U})/\widehat{J}} = \|\widehat{X}(\widehat{I} - \widehat{Q}_N)\|_{C^*(\widehat{S})}.$$

But by the uniqueness property for $\mathcal{O}_n$ [B1] and Proposition 3.2 the right side of both equations are identical. Hence $\sigma$ is an isometry and, in particular, well defined. ∎

COROLLARY 3.4 (Uniqueness variant 2): *Consider the higher rank Cuntz algebra $\mathcal{O}_n = \mathcal{O}_{\{n_1,n_2,\ldots,n_d\}}$ [B1] ($d \le \infty$) represented on $\mathcal{H}$ and suppose that there exists only one operator $U \in B(\mathcal{H})$ satisfying* (A) *and suppose $U$ is unitary. (This setting is the case if and only if $P_{\min(\infty)} = P_{\max(\infty)} = 0$.)*

*Suppose the same fact for a further representation on $\widehat{\mathcal{H}}$. Then the canonical map $\sigma$: $\widetilde{\mathcal{O}}_n \cong C^*(S,U) \to C^*(\widehat{S},\widehat{U})$ is a well defined $*$-isomorphism.*

*Proof:* Since $U$ is the only operator satisfying (A) it must be the operator of Proposition 2.4 with properties (A)–(D). But $U$ is unitary and properties (C)–(D) yield $P_{\min(\infty)} = P_{\max(\infty)} = 0$. The canonical map $\sigma$ follows then by Theorem 3.3 since both ideals $J, \widehat{J}$ are zero.

(On the other hand, if $P_{\min(\infty)} = P_{\max(\infty)} = 0$ then $U$ is unique by Lemma 2.3, hence satisfies (A)-(D) by Proposition 2.4 and is therefore unitary by (C)–(D).) ∎

*Example 3.5:* Consider the higher rank Cuntz algebra $\mathcal{O}_n = \mathcal{O}_{\{n_1,n_2,\ldots,n_d\}}$ represented on $\ell^2(\mathbb{Z})$ by the isometries $S_{i,j}(\delta_k) = \delta_{n_i k+j}$ where $1 \le i \le d, 0 \le j < n_i$. Then one can compute that $P_{\max(N)}$, $N \in \mathbb{Z}_+^d$, projects onto the coordinates $\delta_{\mathbb{Z}n^N-1}$ and consequently $P_{\max(\infty)}$ projects onto $\delta_{-1}$.

(This representation is permutative [BJ], section 2, and if we denote $\sigma_{i,j}(k) = n_i k+j$ then for each $i$ the set $\{\sigma_{i,0},\ldots,\sigma_{i,n_i-1}\}$ forms a multiplicity free branching function system [BJ], Definition 2.1.)

Then the partial isometry $U(\delta_{-1}) = 0$ and $U(\delta_k) = \delta_{k+1}$, $k \in \mathbb{Z}\backslash\{-1\}$, satisfies the rules (A) and (C) and consequently (A)–(D) by 2.3 and 2.4.

Contrarily, the unitary $\widetilde{U}(\delta_k) = \delta_{k+1}$ ($k \in \mathbb{Z}$) satisfies (A) but not (C).

*Example 3.6:* This example refers to Corollary 3.4. Consider the classical Cuntz algebra $\mathcal{O}_2$ represented on $\ell^2(\mathbb{N}_0)$ by the isometries $S_0, S_1$ sketched in the following table.

| $\delta_k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ... |
|---|---|---|---|---|---|---|---|---|
| $S_0(\delta_k)$ | 1 | 3 | 4 | 6 | 8 | 10 | 12 | ... |
| $S_1(\delta_k)$ | 2 | 0 | 5 | 7 | 9 | 11 | 13 | ... |
| $U(\delta_k)$ | 4 | 2 | 8 | 0 | 5 | 16 | 7 | ... |

Then we have $P_{\max(\infty)} = P_{\min(\infty)} = 0$ and only one operator $U$ satisfies (A) and $U$ is unitary; see Corollary 3.4.

## 4. An open question

We have seen by Corollary 3.4 that a unitary operator $U$ satisfying (A), as long as it is the only operator fulfilling (A), generates $\widetilde{\mathcal{O}}_n$ canonically together with the isometries $S_{i,j}$.

If we could abandon the uniqueness of $U$ then we would have a familiar uniqueness, the uniqueness determined by generators and relations. However, we can neither prove nor falsify whether a unitary $U$ together with the isometries $S_{i,j}$ always yields $\widetilde{\mathcal{O}}_n$; see, for example, Example 3.5 where a unitary $\widetilde{U}$ is not the only operator with property (A).

Towards solving this problem we finally would like to present the following minor results. The first result Lemma 4.1 tells us — in combination with Lemma 4.2 — that if a unitary $U$ satisfies (A) then we can map $C^*(S,U)$ canonically to $\widetilde{\mathcal{O}}_n$.

So solving the above question comes to showing that this map is *isometric*.

LEMMA 4.1: *Consider $\mathcal{O}_n = \mathcal{O}_{\{n_1,n_2,\ldots,n_d\}}$ $(d \leq \infty)$ represented on $\mathcal{H}$ and suppose $U \in B(\mathcal{H})$ satisfies* (A)–(B). *Then the canonical map $\sigma\colon C^*(S,U) \to \widetilde{\mathcal{O}}_n$ is well defined.*

*Proof:* Let $X \in \mathrm{Alg}^*(S,U)$. Although the setting of Proposition 3.2 does not seem to be satisfied completely, we nevertheless can apply it to the point that we can choose $Q_N \in \mathrm{Alg}^*(S)$ such that $X(I - Q_N) \in \mathrm{Alg}^*(S)$. The reason is that in the proof of Proposition 3.2 this fact follows already from the properties (A)–(B) which we assume.

The same fact is also true in the image $\sigma$ and it yields the same canonically translated corresponding expression $\sigma(X(I - Q_N))$ in the letters of $S_{i,j}$, since we have the rules (A)–(B) also there.

Note that $\sigma|\mathcal{O}_n$ is an isometry by [B1], Theorem 2.8, wherefore the boundedness of $\sigma$ follows from Proposition 3.2 by

$$\begin{aligned}\|\sigma(X)\| &= \lim_{N\in\mathbb{Z}_+^d} \|\sigma(X(I-Q_N))\|_{\mathcal{O}_n} \\ &= \lim_{N\in\mathbb{Z}_+^d} \|X(I-Q_N)\|_{C^*(S)} \leq \|X\|_{C^*(S,U)}. \quad \blacksquare\end{aligned}$$

Lemma 4.2: *Let $U$ be a unitary satisfying* (A). *Then $U$ fulfills also* (B).

*Proof:* Let $U_0$ be the unique operator with properties (A)–(D) according to Proposition 2.4. By property (C) and Lemma 2.3 we have

$$U_0 = U_0(I - P_{\max(\infty)}) = U(I - P_{\max(\infty)})$$

wherefore $U = U_0 + D$ for the partial isometry $D = UP_{\max(\infty)}$. The support and range projections of $D$ are $P_{\max(\infty)}$ and $P_{\min(\infty)}$, respectively, by properties (C)–(D).

Now for $j \neq 0$ we have $U^*S_{i,j} = U^*US_{i,j-1} = S_{i,j-1}$ which shows (B) in that case.

It is easily computed that

$$\begin{aligned} DD^*S_{i,0} &= P_{\min(\infty)}S_{i,0} = S_{i,0}P_{\min(\infty)} = S_{i,0}DD^*, \\ D^*DS_{i,n_i-1} &= P_{\max(\infty)}S_{i,n_i-1} = S_{i,n_i-1}P_{\max(\infty)} = S_{i,n_i-1}D^*D. \end{aligned}$$

By property (A)

$$DS_{i,n_i-1} = (U - U_0)S_{i,n_i-1} = S_{i,0}(U - U_0) = S_{i,0}D.$$

If we use these formulas then we obtain

$$D^*S_{i,0} = D^*DD^*S_{i,0} = D^*S_{i,0}DD^* = D^*DS_{i,n_i-1}D^* = S_{i,n_i-1}D^*.$$

So the still open equivalence $U^*S_{i,0} = S_{i,n_i-1}U^*$ is now immediate. $\blacksquare$

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